JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 24, No. 2, March–April 2001

Classification of Algorithms for Angular Velocity Estimation

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Scanning the known algorithms for deriving the angular velocity vector from either vector measurements or attitude measurements, it is realized that the algorithms are divided into two categories, each category employing a different approach. One category requires differentiation of the measurements, and one does not. A superficial inspection of the two categories may lead to the conclusion that the two are not related to one another; however, a deeper examination reveals that the two are closely related. In this paper the two categories are formulated in general terms, which enables the presentation of the connection between them. The connection is demonstrated through examples.

Introduction

THE problem of angular-rate estimation of gyro-less spacecraft (SC) is a significant research area lately as new small lightweight SC have been designed and launched. Moreover, even gyro-equipped SC need to use estimated angular rate if, as a result of tumbling, or other undesired maneuvers, its rates exceed the gyro measurement range.

Several ways to obtain the angular rate in a gyro-less SC have been introduced in the last few years. When the attitude is known, one can differentiate the attitude in whatever parameters it is given[1] and use the kinematics equation that connects the derivative of the attitude with the satellite angular rate in order to compute the latter.[2–5] We name this approach the *derivative approach*. Because the differentiation introduces high-frequency noise, some filtering is needed to suppress that noise. The filter can be either a passive low-pass filter[3] or an active filter like an extended Kalman filter (EKF), extended interlaced Kalman filter,[6] state-dependent algebraic Riccati equation (SDARE) filter,[7] pseudolinear Kalman (PSELIKA) filter,[8] and so on.

Another approach avoids differentiation of any kind by applying an estimator directly to the raw measurements. The kinematics equation, which relates the attitude parameters to the angular rate, is simply used as a part of the estimator.[9–11] We name this approach the *estimation approach*.

Most of the known methods for computing the angular velocity vector belong to either one of the two approaches. On one hand, the two approaches seem to be very different from one another, yet both use the same inputs and yield the same outputs; so, there must be a certain connection between the two. It is the purpose of this paper to connect between the two approaches, show where they stem from, expose their common denominator, and, on the other hand, show how they differ. In this paper we do not intend to examine any particular rate determination algorithm in depth or recommend its use, and the sole purpose of the examples that are presented in this paper is, as stated herein, to show the connection between the two approaches.

Analysis

To demonstrate the correspondence between the two approaches, we start with an example. Let D denote the transformation matrix from some reference to body coordinate system, and let $\boldsymbol{\omega}$ denote the vector of the rate of turn of the body with respect to the reference coordinates resolved in body coordinates. Then[12]

$$\dot{D} = -[\omega\times]D \tag{1}$$

where $[\omega\times]$ is the cross-product matrix of $\boldsymbol{\omega}$ defined as follows[13]:

$$[\omega\times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{2}$$

Define the columns of D as three column matrices, that is,

$$D = [\boldsymbol{d}_1 \,|\, \boldsymbol{d}_2 \,|\, \boldsymbol{d}_3] \tag{3}$$

then Eq. (1) can now be written as

$$[\dot{\boldsymbol{d}}_1 \,|\, \dot{\boldsymbol{d}}_2 \,|\, \dot{\boldsymbol{d}}_3] = -[\omega\times][\boldsymbol{d}_1 \,|\, \boldsymbol{d}_2 \,|\, \boldsymbol{d}_3] \tag{4}$$

It is easy to see now that

$$[\dot{\boldsymbol{d}}_1 \,|\, \dot{\boldsymbol{d}}_2 \,|\, \dot{\boldsymbol{d}}_3] = [[\boldsymbol{d}_1\times] \,|\, [\boldsymbol{d}_2\times] \,|\, [\boldsymbol{d}_3\times]]\boldsymbol{\omega} \tag{5}$$

or

$$[\dot{\boldsymbol{d}}_i] = [\boldsymbol{d}_i\times]\boldsymbol{\omega}, \qquad i = 1, 2, 3 \tag{6}$$

Define the column vector $\boldsymbol{s}$ and the matrix $\boldsymbol{D}$ as follows:

$$\boldsymbol{s}^T = \left[\boldsymbol{d}_1^T, \boldsymbol{d}_2^T, \boldsymbol{d}_3^T\right] \tag{7}$$

(where T denotes the transpose) and

$$\boldsymbol{D} = \begin{bmatrix} [\boldsymbol{d}_1\times] \\ [\boldsymbol{d}_2\times] \\ [\boldsymbol{d}_3\times] \end{bmatrix} \tag{8}$$

where $\boldsymbol{s} \in \Re^{9\times 1}$ and $\boldsymbol{D} \in \Re^{9\times 3}$, then Eqs. (6) can be written as

$$\dot{\boldsymbol{s}} = \boldsymbol{D}\boldsymbol{\omega} \tag{9}$$

Derivative Approach

Compute the pseudoinverse of $\boldsymbol{D}$ as follows:

$$\boldsymbol{D}^{\#} = (\boldsymbol{D}^T\boldsymbol{D})^{-1}\boldsymbol{D}^T \tag{10}$$

then $\hat{\boldsymbol{\omega}}$, an estimate of $\boldsymbol{\omega}$, can be computed using the derivative of $\boldsymbol{s}$ as follows:

$$\hat{\boldsymbol{\omega}} = \boldsymbol{D}^{\#}\dot{\boldsymbol{s}} \tag{11}$$

This way of computing $\hat{\boldsymbol{\omega}}$ belongs to the derivative approach.

We cast now the preceding operations in a general form as follows. Let $\boldsymbol{U}$ define a known set of attitude parameters, which define the attitude of a Cartesian coordinate system with respect to a reference one. In the example $\boldsymbol{U}$ represents D, the transformation matrix between the two coordinate systems. We denote by $\boldsymbol{F}$ the operator that acts on $\boldsymbol{U}$ to transform it to $\boldsymbol{s}$. In the example it is the operator described in Eq. (7) that transforms D to $\boldsymbol{s}$; namely, in the example it creates a column vector, which consists of the columns of D put

Received 7 June 1999; revision received 27 April 2000; accepted for publication 31 July 2000.

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one under the other. Finally let $\boldsymbol{f}$ denote the function that describes the dependence of $\dot{s}$ on s and ω. Using the derivative approach $\hat{\omega}$ is extracted from $\dot{s}$ and s using $f^{\#}$, the pseudoinverse function f. In the example this is done in Eq. (11). The general formulation of the derivative approach is summarized as follows:

$$s = F(U) \tag{12}$$

$$\dot{s} = f(s, \omega) \tag{13}$$

$$\hat{\omega} = f^{\#}(\dot{s}, s) \tag{14}$$

Obviously when applying the derivative approach, one needs to differentiate s in order to obtain $\dot{s}$ needed to compute $\hat{\omega}$ according to Eq. (14). Because of the differentiation, the computed $\hat{\omega}$ is contaminated by high-frequency noise, which has to be filtered out by either a passive low-pass filter[3] of the form $G(s) = (1/\tau)/(s + 1/\tau)$ or by an active filter like a Kalman filter[8] (KF). Even if a KF is used to filter out the high-frequency noise, this approach is still the derivative approach.

A block diagram representation of the derivative approach is presented in Fig. 1. Figure 2 presents a typical example of the noisy estimate $\hat{\omega}$, which is obtained after using the pseudoinverse. The data, which were used to test this approach, were real measurements downloaded from the Rossi X-Ray Timing Explorer satellite, which was launched on 30 December 1995. We chose a segment of data starting 4 January 1996 at 21 h, 30 min, and 1.148 s. The attitude was determined using data received from an onboard three-axes magnetometer and a sun sensor. The data were quantized by the telemetry. The magnetometer data were quantized at 0.3 mG/count and the sun sensor at 15 arc-sec/count. The truth rate was obtained from the onboard calibrated Kearfott SKIRU-DII gyros. In Fig. 3 we present $\hat{\omega}_f$, which is the estimated angular rate after applying a low-pass filter whose time constant τ was equal to 40 s. It can be seen that the filtration of the high-frequency noise comes at the expense of a time delay. In Fig. 4 we present the resultant angular velocity rate $\hat{\omega}_f$ obtained when a KF was applied to $\hat{\omega}$. To apply a KF, one needs to know the dynamics, which describes the time propagation of the angular rate vector ω. If we know the parameters of the SC angular dynamics differential equation, and $\boldsymbol{h}$ and $\boldsymbol{T}$ (where $\boldsymbol{h}$ is the momentum of the momentum wheels and $\boldsymbol{T}$ is the external torque operating on the SC), then the following angular dynamics model is well defined:

$$\dot{\omega} = g(\omega, \dot{h}, h, T) \tag{15}$$

The corresponding measurement equation is then

$$\hat{\omega} = I_3\omega + \nu \tag{16}$$

where $\hat{\omega}$ is as shown in Fig. 1, I_3 is the 3×3 identity matrix, and ν is a white-noise vector. In reality the latter is of course not white, but its real nature is very cumbersome, if not impossible, to compute. However the assumption that it is a white-noise vector yields satisfactory results. This is because we are not interested in the optimality

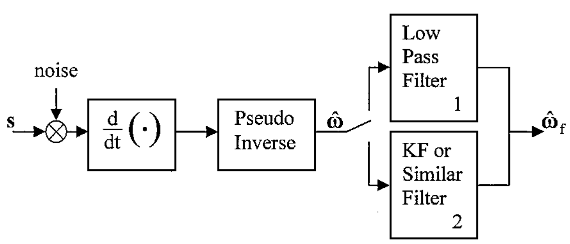

Fig. 1 Block-diagram representation of the derivative approach.

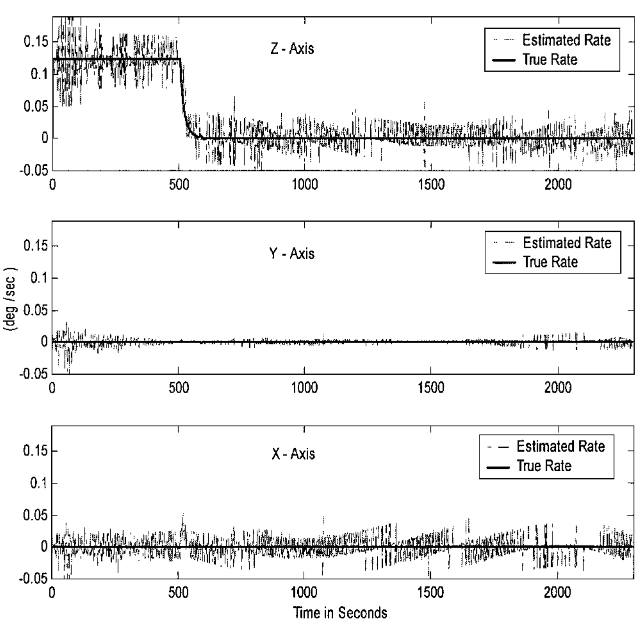

Fig. 2 True angular rate and the estimated angular rate $\hat{\omega}$.

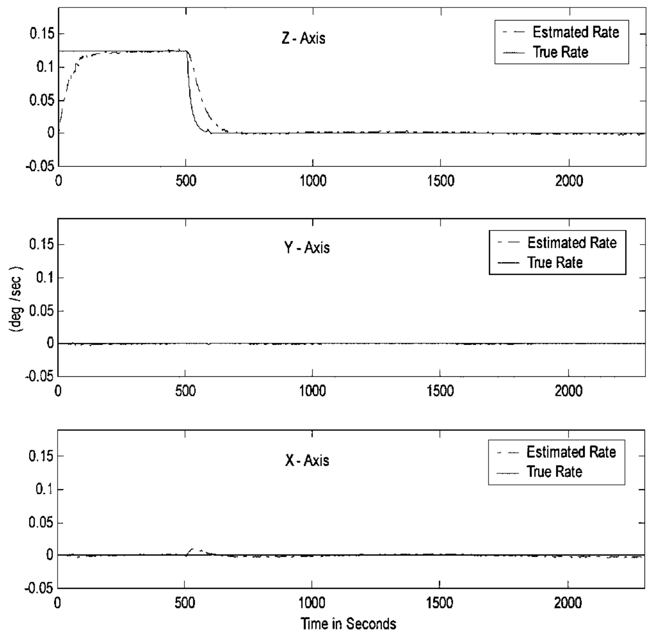

Fig. 3 True angular rate and the low-pass filtered angular rate.

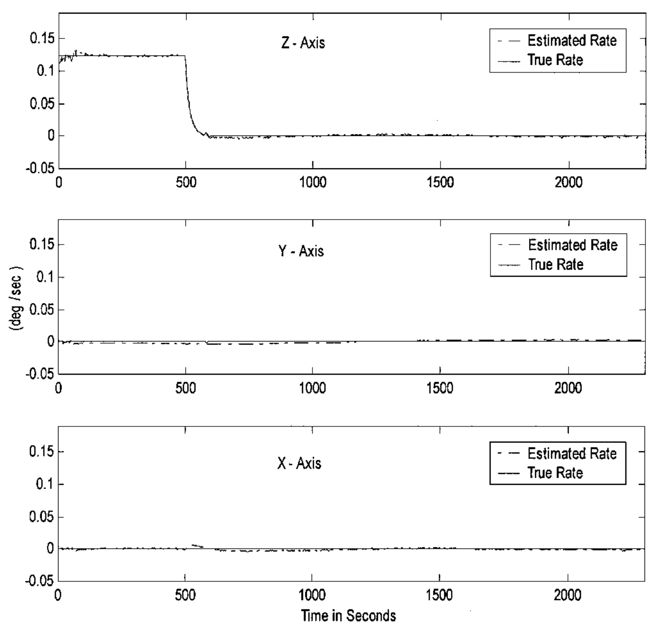

Fig. 4 True angular rate and the KF-filtered angular rate.

of the filter. The only reason some kind of an active filter is sought is that, as seen in Fig. 1, a low-pass filter introduces a considerable delay. On the other hand, an active filter, like a KF, uses the dynamics model to predict the estimate between attitude measurements. Therefore, using $\boldsymbol{T}$ and $\boldsymbol{h}$ it can better track sudden changes. When the measurements are available at a high enough frequency, the dynamics model given in Eq. (15) can be largely simplified by using a first-order Markov model instead of the exact dynamics model.[10] As for the KF, one can use any suitable estimator to estimate $\boldsymbol{\omega}$. For example, one can use the EKF, PSELIKA, or the SDARE filter.[7]

Estimation Approach

Using the example, we now demonstrate the methodology of the estimation approach. We can augment the system model with Eq. (13) to obtain the following *augmented dynamics model*:

$$\begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}(\boldsymbol{\omega}, \dot{\boldsymbol{h}}, \boldsymbol{h}, \boldsymbol{T}) \\ \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{\omega}) \end{bmatrix} \tag{17}$$

Because $\boldsymbol{s}$ is measured, we can write the following measurement equation, which relates the measurement to the augmented state vector

$$\boldsymbol{s}_m = [0 \quad I_n] \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{s} \end{bmatrix} \tag{18}$$

where m denotes a measured quantity and the subscript n is the dimension of $\boldsymbol{s}$.

With the model of Eq. (17) as a dynamics model and that of Eq. (18) as a measurement model, we can apply some estimation algorithm to estimate the augmented state vector $\boldsymbol{x}$ defined as $\boldsymbol{x}^T = [\boldsymbol{\omega}^T, \boldsymbol{s}^T]$. As a result, we obtain an estimate of the angular velocity vector and a slightly improved value for the attitude vector $\boldsymbol{s}$. (Normally we need to add suitable process noise to the dynamics equation and measurement noise to the measurement equation).

The general dynamics equation expressed in Eq. (15) is usually the following SC angular dynamics equation

$$\dot{\boldsymbol{\omega}} = I^{-1}[(I\boldsymbol{\omega} + \boldsymbol{h})\times]\boldsymbol{\omega} + I^{-1}(\boldsymbol{T} - \dot{\boldsymbol{h}}) \tag{19}$$

where I is the SC inertia tensor and $[(I\boldsymbol{\omega} + \boldsymbol{h})\times]$ is the cross-product matrix of the vector $(I\boldsymbol{\omega} + \boldsymbol{h})$. Using Eqs. (9) and (19), the general models of Eqs. (17) and (18) become in this example

$$\begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} I^{-1}[(I\boldsymbol{\omega} + \boldsymbol{h})\times] & 0 \\ \boldsymbol{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{s} \end{bmatrix} + \begin{bmatrix} I^{-1}(\boldsymbol{T} - \dot{\boldsymbol{h}}) \\ 0 \end{bmatrix} \tag{20}$$

$$\boldsymbol{s} = [0_{9\times 3} \quad I_9] \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{s} \end{bmatrix} \tag{21}$$

however, the particular estimator (filter) that is used to perform the estimation is irrelevant. One can use any suitable estimator to estimate $\boldsymbol{\omega}$. For example, one can use the EKF, the PSELIKA, or the SDARE filter.[8] The point is that using the estimation approach the need for differentiation is eliminated. A block diagram representing the essential elements in this approach is presented in Fig. 5.

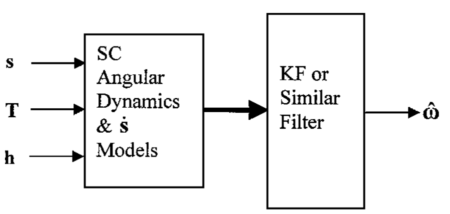

Fig. 5 Block-diagram representation of the estimation approach.

Relation Between the Two Approaches

Both the derivative approach and the estimation approach are based on one key equation that relates the attitude, expressed by a certain parametrization, the time derivative of this attitude parametrization, and the angular rate. The general form of the equation is given in Eq. (13); namely,

$$\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{\omega}) \tag{13}$$

If $\boldsymbol{s}$ and $\dot{\boldsymbol{s}}$ are known and if $\boldsymbol{f}$ has an inverse function $\boldsymbol{f}^{-1}$, an exact value for the angular velocity can be obtained using

$$\boldsymbol{\omega} = \boldsymbol{f}^{-1}(\dot{\boldsymbol{s}}, \boldsymbol{s}) \tag{22}$$

but because we only have $\boldsymbol{s}$ and not $\dot{\boldsymbol{s}}$, we have to apply numerical differentiation to find an approximation to $\dot{\boldsymbol{s}}$. Moreover, usually the function $\boldsymbol{f}$ does not have an inverse function $\boldsymbol{f}^{-1}$; therefore, we have to compute a pseudoinverse function $\boldsymbol{f}^{\#}$ and use it to find $\boldsymbol{\omega}$. Because the numerical differentiation is not accurate and because we use $\boldsymbol{f}^{\#}$ rather than $\boldsymbol{f}^{-1}$, we do not obtain $\boldsymbol{\omega}$ but rather $\hat{\boldsymbol{\omega}}$, an estimate of $\boldsymbol{\omega}$. In other words, we compute an estimated rate vector using

$$\hat{\boldsymbol{\omega}} = \boldsymbol{f}^{\#}(\dot{\boldsymbol{s}}, \boldsymbol{s}) \tag{14}$$

This is basically the principle behind the derivative approach. (As we saw, this approach yields a noisy estimate, which we have to filter out by either a passive low-pass filter that introduces delay, or by an active filter, like a KF, that is applied to the estimated rate. However these are just technical details, associated with this approach. They are not a part of the basis on which this approach is founded).

When using the estimation approach, we still use Eq. (13); however, we do not differentiate $\boldsymbol{s}$, and we do not use the inverse function of $\boldsymbol{f}$. Rather, we augment Eq. (13) with the SC dynamics equation and use our knowledge of the attitude as an input to an estimator of some kind. The state vector of the SC dynamics equation is the angular rate $\boldsymbol{\omega}$; therefore, the augmented state vector is $\boldsymbol{x}^T = [\boldsymbol{\omega}^T, \boldsymbol{s}^T]$. The estimator uses the augmented dynamics model to yield an estimate of the state vector. Because $\boldsymbol{\omega}$ is a part of the augmented state vector, we obtain this way an estimate of $\boldsymbol{\omega}$ through estimation and not through differentiation.

In conclusion, as we pointed out before, both the derivative approach and the estimation approach are based on Eq. (13), which relates the attitude, its time derivative, and the angular rate. This consequence is illustrated further using two more examples of a typical angular-rate determination problem.

Examples

Example 1

In the next example we consider the N vectors described hereafter. Let $\boldsymbol{r}_i$ denote some abstract ith vector as expressed in the reference coordinate system, and let $\boldsymbol{b}_i$ denote the same vector when expressed in the body coordinates. From the laws of dynamics, it is known that

$$D\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{b}}_i + \boldsymbol{\omega} \times \boldsymbol{b}_i \tag{23}$$

where $\dot{\boldsymbol{r}}_i$ is the time derivative of $\boldsymbol{r}_i$ as seen by an observer in the reference coordinates and $\dot{\boldsymbol{b}}_i$ is the time derivative of $\boldsymbol{b}_i$ as seen by an observer in body coordinates. The vector $\boldsymbol{b}_i$ is a measured vector, and $\dot{\boldsymbol{b}}_i$ results from its differentiation. We can write Eq. (23) as follows:

$$\dot{\boldsymbol{b}}_i - D\dot{\boldsymbol{r}}_i = [\boldsymbol{b}_i\times]\boldsymbol{\omega} \tag{24}$$

Note that $\dot{\boldsymbol{r}}_i$ is computable because $\boldsymbol{r}_i$ is usually known because generally the vector is a direction to a certain known planet whose location is given in an almanac, or, like with magnetometer measurements, the vector can be computed using a model. Moreover, quite often the rate of change of $\boldsymbol{r}_i$ is so small that $\dot{\boldsymbol{r}}_i$ is negligible. Let

$$\boldsymbol{s}_i = \boldsymbol{b}_i \tag{25a}$$

then

$$\dot{\boldsymbol{s}}_i = \dot{\boldsymbol{b}}_i \tag{25b}$$

We can augment all N measurements into

$$\begin{bmatrix} \dot{s}_1 \\ \vdots \\ \dot{s}_N \end{bmatrix} = \begin{bmatrix} [b_1 \times] \\ \vdots \\ [b_N \times] \end{bmatrix} \omega \tag{26}$$

Define

$$s = \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} \tag{27a}$$

and

$$B = \begin{bmatrix} [b_1 \times] \\ \vdots \\ [b_N \times] \end{bmatrix} \tag{27b}$$

and compute the pseudoinverse matrix

$$B^{\#} = (B^T B)^{-1} B^T \tag{28}$$

then using Eqs. (25) and (26) we can write

$$\hat{\omega} = B^{\#} \dot{s} \tag{29}$$

where, as before, $\hat{\omega}$ is an estimate of the angular-rate vector. (Note that $B^T B$ is invertible as long as at least two of the vectors are not colinear). Obviously, this estimate is derived applying the derivative approach. Here the differentiated parameter is not strictly attitude but rather a set of vectors resolved in two coordinate systems. As two such vectors determine attitude, these vectors are tantamount to attitude.

Applying this algorithm to the RXTE data mentioned before, we obtained $\hat{\omega}$, which resembled the angular-rate estimate presented in Fig. 2. The application of a low-pass filter with $\tau = 40$ s to this $\hat{\omega}$ resulted in $\hat{\omega}_f$ similar to that shown in Fig. 3.

To apply the estimation approach in this example, we use Eqs. (26) and Eq. (25) to write

$$\dot{s} = B\omega \tag{30}$$

Following Eq. (17), we augment the last equation with the dynamics model of Eq. (19) to obtain

$$\begin{bmatrix} \dot{\omega} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} I^{-1}[(I\omega + h)\times] & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} \omega \\ s \end{bmatrix} + \begin{bmatrix} I^{-1}(T - \dot{h}) \\ 0 \end{bmatrix} \tag{31}$$

The corresponding measurement equation is

$$s = [0_{3N \times 3} \quad I_{3N}] \begin{bmatrix} \omega \\ s \end{bmatrix} \tag{32}$$

which corresponds to Eq. (18). This formulation is correct only when indeed $\dot{r}_i \approx 0$. However, even if $\dot{r}_i \neq 0$ we can use the estimation approach as follows. We can use Eqs. (24), (26), and (27) to write

$$\dot{s} = B\omega + U \tag{33}$$

where

$$U = \begin{bmatrix} D\dot{r}_1 \\ \vdots \\ D\dot{r}_N \end{bmatrix} \tag{34}$$

In this case the augmented system becomes

$$\begin{bmatrix} \dot{\omega} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} I^{-1}[(I\omega + h)\times] & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} \omega \\ s \end{bmatrix} + \begin{bmatrix} I^{-1}(T - \dot{h}) \\ U \end{bmatrix} \tag{35}$$

When using Eqs. (34) and (31) in a PSELIKA filter,[8] which was applied to the RXTE data mentioned before, an $\hat{\omega}_f$ similar to that presented in Fig. 4 was obtained.

Example 2

Let the attitude be parametrized by the attitude quaternion denoted by q; thus, $U = q$ and also $s = q$; therefore, F is a transformation by the unity matrix. The realization of Eq. (13) in this example is

$$\dot{s} = \tfrac{1}{2} Q\omega \tag{36}$$

where

$$Q = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{37}$$

Define the pseudoinverse

$$Q^{\#} = (Q^T Q)^{-1} Q^T \tag{38}$$

and note that

$$Q^T Q = I_3 \tag{39}$$

therefore,

$$Q^{\#} = Q^T \tag{40}$$

From Eqs. (36) and (40) it is easily seen that an estimate of the angular-rate vector can be computed as follows:

$$\hat{\omega} = 2Q^T \dot{s} \tag{41}$$

The general representation that suits Eq. (41) is Eq. (14), which is

$$\hat{\omega} = f^{\#}(\dot{s}, s) \tag{14}$$

and the approach used to obtain $\hat{\omega}$ is, of course, the derivative approach. Plots of the three components of $\hat{\omega}$ look exactly like those presented in Fig. 2, and when passed through a low-pass filter with $\tau = 40$ s they look like those presented in Fig. 3.

Applying the estimation approach, we augment Eqs. (36) and (19) to form

$$\begin{bmatrix} \dot{\omega} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} I^{-1}[(I\omega + h)\times] & 0 \\ \tfrac{1}{2}Q & 0 \end{bmatrix} \begin{bmatrix} \omega \\ s \end{bmatrix} + \begin{bmatrix} I^{-1}(T - \dot{h}) \\ 0 \end{bmatrix} \tag{42}$$

The latter equation is suitable for the application of either one of the filters described before. Equation (42) suits the general form of Eq. (17), and the measurement equation for this example, which is

$$s = [0_{4 \times 3} \quad I_4] \begin{bmatrix} \omega \\ s \end{bmatrix} \tag{43}$$

suits Eq. (18). Plots of the components of the estimated angular velocity $\hat{\omega}$, obtained when using a PSELIKA filter, with the dynamics equation and measurement equation given respectively by Eqs. (42) and (43), look like the plots presented in Fig. 4.

Summary

The derivative approach and the estimation approach for computing the angular velocity of a SC are summarized in Table 1, where U

Table 1 Summary of the two approaches

Basic relations	Two approaches: Derivative approach	Two approaches: Estimation approach
$s = F(U)$ (12)		$\begin{bmatrix} \dot{\omega} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} g(\omega, \dot{h}, h, T) \\ f(s, \omega) \end{bmatrix}$ (17)
	$\hat{\omega} = f^{\#}(\dot{s}, s)$ (14)	
$\dot{s} = f(s, \omega)$ (13)		$s = [0 \quad I_n] \begin{bmatrix} \omega \\ s \end{bmatrix}$ (18)

represents attitude information, s represents the attitude information in a different format, and F is the operator, which transforms U to s. The latter conversion is used when the attitude is represented by the direction cosine matrix but is not needed when other parameters, like Euler angles or quaternions, are used to represent the attitude. The time derivative of s is related to s itself and to the angular velocity ω. The functional relationship between these three vectors is denoted by f and is presented in Eq. (13). As shown in Eq. (14), the derivative approach uses the function $f^{\#}$, which is the pseudoinverse function of f, to compute an estimate of ω based on s and its derivative, which is obtained by numerical differentiation. The latter operation usually introduces a large amount of high-frequency noise, which has to be removed by either a passive low-pass filter or by some kind of an active filter like the KF. This filter should not be confused with the filter used in the estimation approach, which is based on the augmentation of the basic relation shown in Eq. (13), that is, on f and the angular dynamics equation of the SC. The augmented model shown in Eq. (17) is a state-space model, which calls for the application of an estimator to estimate the state vector x, where $x^T = [\omega^T, s^T]$. The suitable measurement model shown in Eq. (18), which is needed for the estimation process, simply relates the measured (or computed) attitude parameter s to the state vector. An estimate of x yields, of course, the desired estimate of ω.

In our tests with real data, we observed that using the derivative approach we indeed obtained angular-rate estimates with high-frequency noise. When passing the noisy estimates through a low-pass filter, we obtained smooth estimates but at the expense of a delay. The question is then asked: Is this approach acceptable? The answer to this question depends on the application for which the estimate is sought; however, if the supreme criterion is accuracy then the only acceptable choices are either the derivative approach with an active (KF like) filter or the estimation approach. Because visually both yield the same accuracy, which is presented in Fig. 4, then in order to compare the two we need a quantitative comparison. To meet this end, we define the error vector e and a figure of merit (FM) as follows

$$e = \omega - \hat{\omega}_f, \qquad \mathrm{FM} = \left[\overline{e_x^2} + \overline{e_y^2} + \overline{e_z^2}\right]^{\frac{1}{2}}$$

where

$$\overline{e_i^2} = \frac{1}{t_f - t_0}\int_{t_0}^{t_f} e_i^2\,\mathrm{d}t, \qquad i = x, y, z$$

t_0 is the beginning of the time span over which the FM is computed, and t_f is its end. Using the PSELIKA filter to filter the high-frequency noise obtained when the derivative approach is applied to quaternion measurements (see Example 2), the FM has a value of 1.5311×10^{-3} deg/s. When the same kind of filter is used with the same data in the estimation approach, the FM is 6.1839×10^{-4} deg/s. In this case the latter algorithm yields slightly more accurate results. This probably stems from the fact that when using the former algorithm we perform a numerical differentiation, which introduces some error in the signal. At any rate the preceding check has to be done for every application, and based on the results the analyst has to make the choice.

Conclusions

The most common algorithms for estimating the angular rate of a SC belong to one of two categories, each based on a different approach. In this paper we showed that both categories are based on one key differential equation that vary from one attitude parameterization to another. This equation relates the attitude parameterization, its time derivative, and the sought angular rate. According to one approach, which we name derivative approach, the known attitude is differentiated numerically, and a pseudoinverse, which is based on the key differential equation, is applied to the differentiated attitude to yield an estimate of the SC angular rate. According to the other approach, which avoids differentiation altogether, the key differential equation is augmented with the differential equation describing the angular motion of the SC. The augmented dynamics equation is used in some kind of an estimator, which uses the known attitude parameter as measurement.

The two approaches are presented in general terms, and, using several examples, it is shown that indeed the most popular algorithms for angular-rate estimation fall into these two categories that hinge on a common differential equation.

The purpose of this work was not to recommend one approach over the other or describe any algorithm in details, but rather to give a broad outlook at the two approaches and show the interrelations between them.

Acknowledgments

This work was supported by NASA-Goddard Space Flight Center, Grant NAG5-8770. The author wishes to thank Richard R. Harman of the Flight Dynamics Analysis Branch, Code 572, NASA Goddard Space Flight Center for running the examples presented in this paper.

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